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Re-examination of Yang–Lee zeros of the anisotropic Ising models on square, triangular and honeycomb lattices

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Abstract

We have determined the zeros of the partition functions of the anisotropic Ising models on square, triangular and honeycomb lattices in the absence of a magnetic field, with arbitrary combinations of interactions. It is found that the zeros are generally distributed over areas. However, the zeros near the positive real axis are distributed on the unit circle in a complex plane, with the zero density $g(\theta) \sim |\theta|$, which leads to logarithmic singularity of the free energy. Our results generalize Fisher's results for isotropic cases.

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1. Introduction

In 1952, Yang and Lee [1] proposed a general theory of phase transitions. They observed that the grand partition function of a real gas with a hard core in a finite volume is a polynomial in fugacity. They introduced the complex-fugacity zeros and showed that in the thermodynamic limit if the zero distribution approaches the positive real axis, a phase transition arises. They further applied their theory to an Ising ferromagnet and proved the famous circle theorem, which states that the zeros are distributed on the unit circle in the complex-activity plane [2, 3].

In 1964, Fisher [4] observed that the partition function of an Ising model without a magnetic field may be written as a polynomial in a variable that is a function of temperature. He introduced the complex-temperature zeros and showed that the zeros are located on circles for a square lattice [5] and that the logarithmic singularity is solely determined by the zero distribution near the positive real axis. Itzykson *et al* [6, 7] further considered cases with a magnetic field where the partition function may be expressed as a polynomial in two variables. In addition to the Ising model, the complex-temperature zeros of the Potts model have been studied [8].

Some researchers [9–11] have considered the zeros of the partition functions of the two-dimensional anisotropic Ising models in the absence of a magnetic field, with the interaction strengths J_1:J_2 = integer:integer (square lattice) and J_1:J_2:J_3 = integer:integer:integer (triangular lattice), where the partition function may be expressed as a polynomial in a variable that is a function of temperature. In this paper, we will consider cases with arbitrary combinations of interaction strengths where the partition function may be expressed as a polynomial in two variables, w_1 and w_2 (square), or three variables, w_1, w_2 and w_3 (triangular or honeycomb). The zeros are determined.

This paper is organized as follows. In sections 2–4, the zeros of the square, triangular and honeycomb lattices are derived, respectively. In section 5, a summary is given.

2. Square lattice

From Kaufman's exact solution [12], we obtain the partition function on a very large lattice:

$$Z_N^S = 2^{M_1M_2} \prod_{n_1=1}^{M_1} \prod_{n_2=1}^{M_2/2} (C_1C_2 - S_1\cos\phi_1 - S_2\cos\phi_2) \tag{1}$$

where $C_i = \cosh 2K_i$, $S_i = \sinh 2K_i$, $\phi_i = 2\pi n_i/M_i$ and $N = M_1M_2$ is the total number of lattice points.

In the thermodynamic limit, the free energy per site is given by

$$f/k_BT = -\ln 2 - \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi} \mathrm{d}\phi_1\,\mathrm{d}\phi_2 \ln(C_1C_2 - S_1\cos\phi_1 - S_2\cos\phi_2). \tag{2}$$

Let us determine the zeros $\alpha(n_1, n_2)$ of the partition function polynomial in the variable w_1:

$$C_2\sqrt{1+\eta^2} - \eta\cos\phi_1 - S_2\cos\phi_2 = 0. \tag{3}$$

Thus,

$$\begin{aligned}\eta(n_1,n_2) = (\alpha^{-1}-\alpha)/2 &= \frac{1}{S_2^2+\sin^2\phi_1}\Big[S_2\cos\phi_1\cos\phi_2 \\ &\quad \pm \mathrm{i}\sqrt{(\sin^2\phi_1+\sin^2\phi_2)S_2^2 + S_2^4\sin^2\phi_2 + \sin^2\phi_1}\Big] \\ &\equiv r\mathrm{e}^{\mathrm{i}\theta}\end{aligned} \tag{4}$$

where

$$r(n_1,n_2) = \sqrt{\frac{1+S_2^2\sin^2\phi_2}{S_2^2+\sin^2\phi_1}} \tag{5}$$

and

$$\theta(n_1,n_2) = \cos^{-1}\frac{S_2\cos\phi_1\cos\phi_2}{r(S_2^2+\sin^2\phi_1)}. \tag{6}$$

Here $w_i = \exp(-2K_i)$.

Since equation (4) contains two variables ϕ_1 and ϕ_2, we see that the zeros are located in areas, as shown in figure 1. Here $S_2 = 2$. The zero distribution approaches the real axis, giving $S_{1c} = \pm 1/2$, which correspond to a ferromagnetic phase transition of the ferromagnetic Ising model with J_1:$J_2 = \ln(1/2+\sqrt{5}/2)$: $\ln(2+\sqrt{5})$, $J_1 > 0$, $J_2 > 0$, with $T_c = 2J_2/[k_B\ln(2+\sqrt{5})]$, and an antiferromagnetic phase transition of the antiferromagnetic Ising model with $|J_1|$:$J_2 = \ln(1/2+\sqrt{5}/2)$: $\ln(2+\sqrt{5})$, $J_1 < 0$, $J_2 > 0$, with $T_c = 2J_2/[k_B\ln(2+\sqrt{5}]$, respectively.

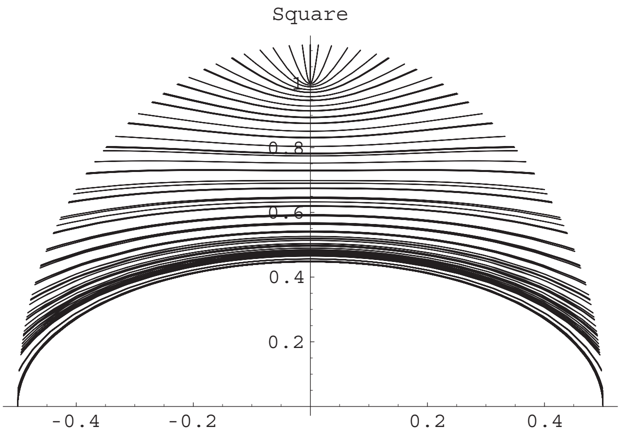

Figure 1. The square-lattice zeros are distributed in the shaded areas in the η-plane. Here $S_2 = 2$.

Using the zero distribution, we obtain

$$f/k_BT = D - \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi} \mathrm{d}\phi_1\,\mathrm{d}\phi_2 \ln(\sinh^2 2K_1 - 2r\sinh 2K_1\cos\theta + r^2)$$
$$= D - \frac{1}{8\pi^2}\int\int \mathrm{d}r\,\mathrm{d}\theta\, g(r,\theta)\ln(\sinh^2 2K_1 - 2r\sinh 2K_1\cos\theta + r^2) \quad (7)$$

where

$$D = -\ln 2 - \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi} \mathrm{d}\phi_1\,\mathrm{d}\phi_2 \ln[(C_2^2 - \cos^2\phi_1)/(C_1C_2 + S_1\cos\phi_1 + S_2\cos\phi_2)] \quad (8)$$

and

$$g(r,\theta) = \left|\frac{\partial(\phi_1,\phi_2)}{\partial(r,\theta)}\right|. \quad (9)$$

Let us consider the ferromagnetic case $J_1 > 0$, $J_2 > 0$. $\phi_1 = \phi_2 = 0$ corresponds to a ferromagnetic critical point. Expanding the integrand in equation (7) around $\phi_1 = \phi_2 = 0$ and retaining the largest terms, we obtain the singular part of the free energy:

$$f_s \sim \int_0^{\phi_{10}}\int_0^{\phi_{20}} \mathrm{d}\phi_1\,\mathrm{d}\phi_2 \ln[(S_1S_2 - 1)^2 + \Gamma(\phi_1,\phi_2)] \quad (10)$$

where ϕ_{10} and ϕ_{20} are small numbers and

$$\Gamma(\phi_1,\phi_2) = (S_1 + S_2)(S_1\phi_1^2 + S_2\phi_2^2). \quad (11)$$

Define $\sqrt{(S_1 + S_2)S_1}\phi_1 = \theta\cos\chi$ and $\sqrt{(S_1 + S_2)S_2}\phi_2 = \theta\sin\chi$ ($\theta \geqslant 0$). Equation (10) becomes

$$f_s \sim \int_0^{\theta_0}\int_0^{\chi_0} \theta\ln[(S_1S_2 - 1)^2 + \theta^2]\,\mathrm{d}\chi\,\mathrm{d}\theta$$
$$\sim \int_0^{\theta_0} \theta\ln[(S_1S_2 - 1)^2 + \theta^2]\,\mathrm{d}\theta$$
$$\sim \int_{-\theta_0}^{\theta_0} |\theta|\ln[(S_1S_2 - 1) + \mathrm{i}\theta]\,\mathrm{d}\theta \quad (12)$$

where $\theta_0 = \sqrt{\Gamma(\phi_{10}, \phi_{20})}$ and $\chi_0 = \tan^{-1}\sqrt{S_2\phi_{20}^2/S_1 c\phi_{10}^2}$. As $T \to T_c$, we have $(S_1 S_2 - 1) \sim t$ and hence $f_s \sim t^2 \ln|t|$.

We find that near the positive real axis, the zeros are located on the unit circle in the X-plane, with the zero density $g(\theta) \sim |\theta|$, which gives the logarithmic singularity. Here $X = \eta S_2$.

We obtain a similar conclusion for the antiferromagnetic case, with $X = \mathrm{sgn}(J_1)\eta|S_2|$.

3. Triangular lattice

The partition function on a large triangular lattice is given by

$$(Z_N^T)^2 = 2^{2M_1M_2} \prod_{n_1=1}^{M_1} \prod_{n_2=1}^{M_2} [C_1C_2C_3 + S_1S_2S_3 - S_1\cos\phi_1 - S_2\cos\phi_2 - S_3\cos(\phi_1+\phi_2)] \quad (13)$$

where $C_i = \cosh 2K_i$, $S_i = \sinh 2K_i$, $\phi_i = 2\pi n_i/M_i$ and $N = M_1M_2$ is the total number of lattice points.

The free energy per site is given by

$$f/k_BT = -\ln 2 - \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi} \mathrm{d}\phi_1\,\mathrm{d}\phi_2 \ln[C_1C_2C_3 + S_1S_2S_3 - S_1\cos\phi_1 - S_2\cos\phi_2 - S_3\cos(\phi_1+\phi_2)]. \quad (14)$$

The critical conditions are given by [13, 14]

$$\omega_1 + \omega_2 + \omega_3 + \omega_4 = 2\max(\omega_1, \omega_2, \omega_3, \omega_4) \quad (15)$$

where $\omega_1 = \exp(K_1 + K_2 + K_3)$, $\omega_2 = \exp(K_3 - K_1 - K_2)$, $\omega_3 = \exp(K_2 - K_3 - K_1)$ and $\omega_4 = \exp(K_1 - K_2 - K_3)$.

Let us determine the zeros $\alpha(n_1, n_2)$ of the partition function polynomial in the variable w_1:

$$C_2C_3\sqrt{1+\eta^2} + \eta S_2S_3 - \eta\cos\phi_1 - S_2\cos\phi_2 - S_3\cos(\phi_1+\phi_2) = 0. \quad (16)$$

Thus,

$$\eta(n_1, n_2) = (\alpha^{-1} - \alpha)/2 = \frac{-A_2 \pm \sqrt{A_2^2 - A_1A_3}}{A_1} \equiv r\mathrm{e}^{\mathrm{i}\theta} \quad (17)$$

where

$$A_1 = C_2^2C_3^2 - (S_2S_3 - \cos\phi_1)^2 \quad (18)$$

$$A_2 = (S_2S_3 - \cos\phi_1)[S_2\cos\phi_2 + S_3\cos(\phi_1+\phi_2)] \quad (19)$$

$$A_3 = C_2^2C_3^2 - [S_2\cos\phi_2 + S_3\cos(\phi_1+\phi_2)]^2 \quad (20)$$

$$r = \sqrt{\frac{A_3}{A_1}} \qquad \cos\theta = -\frac{A_2}{rA_1}. \quad (21)$$

Here $w_i = \exp(-2K_i)$.

Since equation (17) contains two variables ϕ_1 and ϕ_2, we see that the zeros are located in areas, as shown in figure 2. Here $S_2 = 2$ and $S_3 = 3$. The zero distribution approaches the real axis, giving $S_{1c} = -1$ and $S_{1c} = -7$, which correspond to an antiferromagnetic phase transition of the antiferromagnetic Ising model with $|J_1|{:}J_2{:}J_3 = \ln(1+\sqrt{2}){:}\ln(2+\sqrt{5}){:}\ln(3+\sqrt{10})$, $J_1 < 0$, $J_2 > 0$, $J_3 > 0$, with $T_c = 2J_2/[k_B\ln(2+\sqrt{5})]$, and an antiferromagnetic phase transition of the antiferromagnetic Ising model with $|J_1|{:}J_2{:}J_3 = \ln(7+\sqrt{50}){:}\ln(2+\sqrt{5}){:}\ln(3+\sqrt{10})$, $J_1 < 0$, $J_2 > 0$, $J_3 > 0$, with $T_c = 2J_2/[k_B\ln(2+\sqrt{5})]$, respectively.

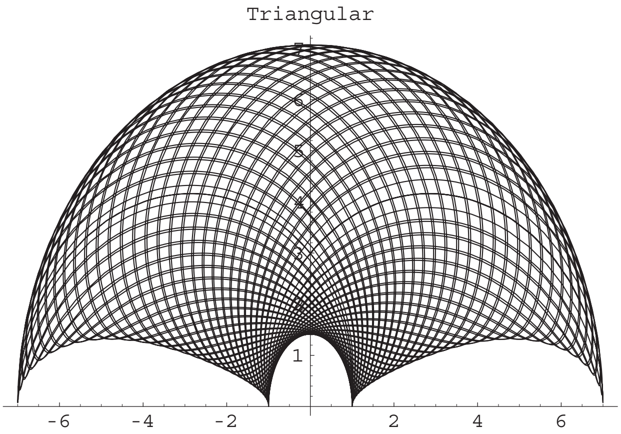

Figure 2. The triangular-lattice zeros are distributed in the shaded region in the η-plane. Here $S_2 = 2$ and $S_3 = 2$.

The free energy per site may be expressed as

$$f/k_BT = D - \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi} \mathrm{d}\phi_1\,\mathrm{d}\phi_2 \ln[\sinh^2 2K_1 - 2r\sinh 2K_1\cos\theta + r^2] \tag{22}$$

where

$$D = -\ln 2 - \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi} \mathrm{d}\phi_1\,\mathrm{d}\phi_2 \ln\{A_1[C_1C_2C_3 - S_1S_2S_3 + S_1\cos\phi_1 + S_2\cos\phi_2 + S_3\cos(\phi_1+\phi_2)]^{-1}\}. \tag{23}$$

In the following, without loss of generality, we let $|J_1| > |J_2| > |J_3|$.

3.1. $J_1 > 0$, $J_2 > 0$

$\phi_1 = \phi_2 = 0$ corresponds to a critical point. Expanding the integrand in equation (22) around $\phi_1 = \phi_2 = 0$ and retaining the largest terms, we obtain the singular part of the free energy:

$$\begin{aligned} f_s &\sim \int_0^{\phi_{10}}\int_0^{\phi_{20}} \mathrm{d}\phi_1\,\mathrm{d}\phi_2 \ln\{[S_1(S_2+S_3) + S_2S_3 - 1]^2 + \Gamma(\phi_1,\phi_2)\} \\ &\sim \int_{-\theta_0}^{\theta_0} |\theta| \ln[S_1(S_2+S_3) + S_2S_3 - 1 + \mathrm{i}\theta]\,\mathrm{d}\theta \end{aligned} \tag{24}$$

where ϕ_{10} and ϕ_{20} are small numbers and

$$\Gamma(\phi_1,\phi_2) = (S_1 + S_2 + S_3 - S_1S_2S_3)[(S_1+S_3)\phi_1^2 + (S_2+S_3)\phi_2^2 + 2S_3\phi_1\phi_2]. \tag{25}$$

As $T \to T_c$, we have $[S_1(S_2+S_3) + S_2S_3 - 1] \sim t$ and hence $f_s \sim t^2 \ln|t|$. We obtain a similar conclusion with $X = \eta(S_2+S_3)/(1 - S_2S_3)$.

3.2. $J_1 < 0$, $J_2 < 0$

$\phi_1 = \phi_2 = \pi$ corresponds to a critical point. Following the same procedure, we obtain

$$\begin{aligned} f_s &\sim \int_{-\phi_{10}}^{\phi_{10}}\int_{-\phi_{20}}^{\phi_{20}} \mathrm{d}\phi_1\,\mathrm{d}\phi_2 \ln\{[-S_1(-S_2+S_3) - S_2S_3 - 1]^2 + \Gamma(\phi_1,\phi_2)\} \\ &\sim \int_{-\theta_0}^{\theta_0} |\theta| \ln[-S_1(-S_2+S_3) - S_2S_3 - 1 + \mathrm{i}\theta)\}\,\mathrm{d}\theta \end{aligned} \tag{26}$$

where ϕ_{10} and ϕ_{20} are small numbers and

$$\Gamma(\phi_1, \phi_2) = (-S_1 - S_2 + S_3 - S_1 S_2 S_3)[(-S_1 + S_3)\phi_1^2 + (-S_2 + S_3)\phi_2^2 + 2S_3\phi_1\phi_2]. \tag{27}$$

We obtain a similar conclusion with $X = \eta(S_2 - S_3)/(1 + S_2 S_3)$.

3.3. $J_1 > 0$, $J_2 < 0$

$\phi_1 = 0$ and $\phi_2 = \pi$ corresponds to a critical point. Following the same procedure, we obtain

$$\begin{aligned} f_s &\sim \int_0^{\phi_{10}} \int_{-\phi_{20}}^{\phi_{20}} \mathrm{d}\phi_1\, \mathrm{d}\phi_2 \ln\{[S_1(-S_2 - S_3) + S_2 S_3 - 1]^2 + \Gamma(\phi_1, \phi_2)\} \\ &\sim \int_{-\theta_0}^{\theta_0} |\theta| \ln[S_1(-S_2 - S_3) + S_2 S_3 - 1 + \mathrm{i}\theta]\, \mathrm{d}\theta \end{aligned} \tag{28}$$

where ϕ_{10} and ϕ_{20} are small numbers and

$$\Gamma(\phi_1, \phi_2) = (S_1 - S_2 - S_3 - S_1 S_2 S_3)[(S_1 - S_3)\phi_1^2 + (-S_2 - S_3)\phi_2^2 - 2S_3\phi_1\phi_2]. \tag{29}$$

We obtain a similar conclusion with $X = -\eta(S_2 + S_3)/(1 - S_2 S_3)$.

3.4. $J_1 < 0$, $J_2 > 0$

$\phi_1 = \pi$ and $\phi_2 = 0$ correspond to a critical point. Following the same procedure, we obtain

$$\begin{aligned} f_s &\sim \int_{-\phi_{10}}^{\phi_{10}} \int_0^{\phi_{20}} \mathrm{d}\phi_1\, \mathrm{d}\phi_2 \ln\{[-S_1(S_2 - S_3) - S_2 S_3 - 1]^2 + \Gamma(\phi_1, \phi_2)\} \\ &\sim \int_{-\theta_0}^{\theta_0} |\theta| \ln[-S_1(S_2 - S_3) - S_2 S_3 - 1 + \mathrm{i}\theta]\, \mathrm{d}\theta \end{aligned} \tag{30}$$

where ϕ_{10} and ϕ_{20} are small numbers and

$$\Gamma(\phi_1, \phi_2) = (-S_1 + S_2 - S_3 - S_1 S_2 S_3)[(-S_1 - S_3)\phi_1^2 + (S_2 - S_3)\phi_2^2 - 2S_3\phi_1\phi_2]. \tag{31}$$

We obtain a similar conclusion with $X = -\eta(S_2 - S_3)/(1 + S_2 S_3)$.

4. Honeycomb lattice

According to the honeycomb–triangular duality, there exists a relation between the partition functions of the ferromagnetic Ising models on the triangular and honeycomb lattices [16]:

$$Z_{2N}^H(L_1, L_2, L_3) = (\tfrac{1}{2} \sinh 2K_1 \sinh 2K_2 \sinh 2K_3)^{-N/2} Z_N^T(K_1, K_2, K_3) \tag{32}$$

where

$$\tanh K_i = \exp(-2L_i) \qquad i = 1, 2, 3. \tag{33}$$

Here $L_i = J_i/k_B T$. Equation (33) implies that $\sinh 2K_i \sinh 2L_i = 1$.

Substituting equation (13) into (32) gives

$$(Z_{2N}^H)^2 = 2^{M_1 M_2} \prod_{n_1=1}^{M_1} \prod_{n_2=1}^{M_2} [C_1 C_2 C_3 + 1 - S_2 S_3 \cos\phi_1 - S_3 S_1 \cos\phi_2 - S_1 S_2 \cos(\phi_1 + \phi_2)] \tag{34}$$

where $C_i = \cosh 2L_i$, $S_i = \sinh 2L_i$, $\phi_i = 2\pi n_i/M_i$ and $N = M_1 M_2$.

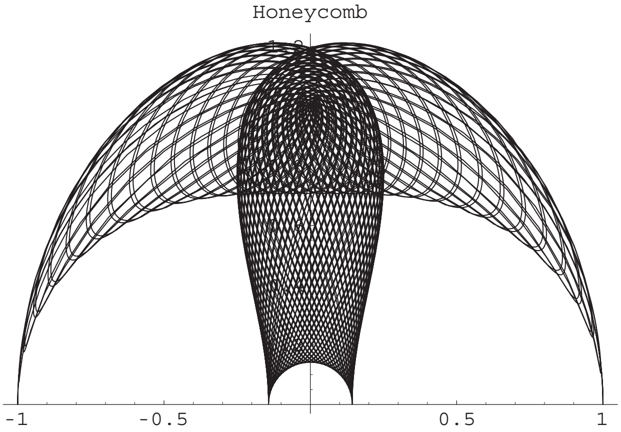

Figure 3. The honeycomb-lattice zeros are distributed in the shaded areas in the η-plane. Here $S_2 = 3$ and $S_3 = 2$. The real zeros $\eta = \pm 1/7$ correspond to pseudo-critical points.

The free energy per site is given by

$$f/k_B T = -\frac{1}{4}\ln 2 - \frac{1}{16\pi^2}\int_0^{2\pi}\int_0^{2\pi} \mathrm{d}\phi_1\,\mathrm{d}\phi_2 \ln[C_1C_2C_3 + 1 - S_2S_3\cos\phi_1 - S_3S_1\cos\phi_2 - S_1S_2\cos(\phi_1+\phi_2)]. \quad (35)$$

Equations (34) and (35) are also valid for the antiferromagnetic cases.

The critical conditions are $\zeta_1\zeta_2\zeta_3 - \zeta_1\zeta_2 - \zeta_2\zeta_3 - \zeta_3\zeta_1 - \zeta_1 - \zeta_2 - \zeta_3 + 1 = 0$. Here $\zeta_i = \exp(-2|L_i|)$ [15].

Let us determine the zeros $\alpha(n_1, n_2)$ of the partition function polynomial in the variable w_1:

$$C_2C_3\sqrt{1+\eta^2} + 1 - S_2S_3\cos\phi_1 - S_3\eta\cos\phi_2 - \eta S_2\cos(\phi_1+\phi_2) = 0. \quad (36)$$

Thus,

$$\eta(n_1, n_2) = (\alpha^{-1} - \alpha)/2 = \frac{-A_2 \pm \sqrt{A_2^2 - A_1A_3}}{A_1} \equiv r\mathrm{e}^{\mathrm{i}\theta} \quad (37)$$

where

$$A_1 = C_2^2C_3^2 - [S_3\cos\phi_2 + S_2\cos(\phi_1+\phi_2)]^2 \quad (38)$$

$$A_2 = (1 - S_2S_3\cos\phi_1)[S_3\cos\phi_2 + S_2\cos(\phi_1+\phi_2)] \quad (39)$$

$$A_3 = C_2^2C_3^2 - (-1 + S_2S_3\cos\phi_1)^2 \quad (40)$$

$$r = \sqrt{\frac{A_3}{A_1}} \qquad \cos\theta = -\frac{A_2}{rA_1}. \quad (41)$$

Here $w_i = \exp(-2L_i)$.

Since equation (37) contains two variables ϕ_1 and ϕ_2, we see that the zeros are located in areas, as shown in figure 3. Here $S_2 = 3$ and $S_3 = 2$. The zero distribution approaches the real axis, giving $S_{1c} = \pm 1$ and $S_{1pc} = \pm 1/7$. $S_{1c} = \pm 1$ correspond to a ferromagnetic phase transition of the ferromagnetic Ising model with $J_1{:}J_2{:}J_3 = \ln(1+\sqrt{2}){:}\ln(3+\sqrt{10}){:}\ln(2+\sqrt{5})$, $J_1 > 0$, $J_2 > 0$, $J_3 > 0$, with $T_c = 2J_3/[k_B\ln(2+\sqrt{5})]$, and an antiferromagnetic phase transition of the antiferromagnetic Ising model with $|J_1|{:}J_2{:}J_3 = \ln(1+\sqrt{2}){:}\ln(3+\sqrt{10}){:}\ln(2+\sqrt{5})$,

$J_1 < 0$, $J_2 > 0$, $J_3 > 0$, with $T_c = 2J_3/[k_B \ln(2+\sqrt{5})]$, respectively. $S_{1pc} = \pm 1/7$ correspond to pseudo-critical points, which arise as a consequence of the negative sign in $\sqrt{1+\eta^2} = \pm|\sqrt{1+\eta^2}|\exp(\mathrm{i}\epsilon)$ in equation (36).

The free energy per site may be expressed as

$$f/k_BT = D - \frac{1}{16\pi^2}\int_0^{2\pi}\int_0^{2\pi} \mathrm{d}\phi_1\,\mathrm{d}\phi_2 \ln[\sinh^2 2L_1 - 2r\sinh L_1\cos\theta + r^2] \tag{42}$$

where

$$D = -\frac{1}{4}\ln 2 - \frac{1}{16\pi^2}\int_0^{2\pi}\int_0^{2\pi} \mathrm{d}\phi_1\,\mathrm{d}\phi_2 \ln\{A_1[C_1C_2C_3 - 1 + S_2S_3\cos\phi_1 + S_3S_1\cos\phi_2 + S_1S_2\cos(\phi_1+\phi_2)]^{-1}\}. \tag{43}$$

Let us consider the cases $J_1 > 0$, $J_2 > 0$, $J_3 > 0$ or $J_1 < 0$, $J_2 < 0$, $J_3 < 0$. $\phi_1 = \phi_2 = 0$ corresponds to a critical point. Expanding the integrand in equation (42) around $\phi_1 = \phi_2 = 0$ and retaining the largest terms, we obtain the singular part of the free energy:

$$\begin{aligned} f_s &\sim \int_0^{\phi_{10}}\int_0^{\phi_{20}} \mathrm{d}\phi_1\,\mathrm{d}\phi_2 \ln\{[S_1(1-S_2S_3) + S_2 + S_3]^2 + \Gamma(\phi_1,\phi_2)\} \\ &\sim \int_{-\theta_0}^{\theta_0} |\theta| \ln\{[S_1(1-S_2S_3) + S_2 + S_3] + \mathrm{i}\theta\}\,\mathrm{d}\theta \end{aligned} \tag{44}$$

where ϕ_{10} and ϕ_{20} are small numbers and

$$\Gamma(\phi_1,\phi_2) = (S_1S_2 + S_2S_3 + S_3S_1 - 1)[S_1S_2(\phi_1+\phi_2)^2 + S_2S_3\phi_1^2 + S_3S_1\phi_2^2]. \tag{45}$$

As $T \to T_c$, we have $[S_1(1-S_2S_3) + S_2 + S_3] \sim t$ and hence $f_s \sim t^2 \ln|t|$.

We obtain a similar conclusion with $X = \mathrm{sgn}(J_1)\eta(|S_2S_3| - 1)/(|S_2| + |S_3|)$.

5. Conclusions

(1) In the anisotropic cases, the zeros are generally located in areas.

(2) Near the positive real axis, the zeros are located on the unit circle in the X-plane. Here X is given by

  (i) Square: $X = \eta\,\mathrm{sgn}(J_1)\sinh 2|K_2|$

  (ii) Triangular ($|J_1| > |J_2| > |J_3|$):

$$X = \eta\frac{\sinh 2K_2 + \sinh 2K_3}{1 - \sinh 2K_2\sinh 2K_3} \qquad (J_1 > 0, J_2 > 0)$$

$$X = -\eta\frac{-\sinh 2K_2 + \sinh 2K_3}{1 + \sinh 2K_2\sinh 2K_3} \qquad (J_1 < 0, J_2 < 0)$$

$$X = \eta\frac{-\sinh 2K_2 - \sinh 2K_3}{1 - \sinh 2K_2\sinh 2K_3} \qquad (J_1 > 0, J_2 < 0)$$

$$X = -\eta\frac{\sinh 2K_2 - \sinh 2K_3}{1 + \sinh 2K_2\sinh 2K_3} \qquad (J_1 < 0, J_2 > 0).$$

  (iii) Honeycomb:

$$X = \mathrm{sgn}(J_1)\eta\frac{\sinh 2|L_2|\sinh 2|L_3| - 1}{\sinh 2|L_2| + \sinh 2|L_3|}.$$

(3) The logarithmic singularity arises as a consequence of the zero distribution near the positive real axis, $g(\theta) \sim |\theta|$.

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